Classification of modules of the intermediate series over Ramond $N = 2$ superconformal algebras[1]

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Abstract

In this paper, we first discuss the structure of the Ramond $N = 2$ superconformal algebras. Then we classify the modules of the intermediate series over Ramond $N = 2$ superconformal algebra.

1. Introduction

More than two decades ago, superconformal algebras were first constructed independently and almost at the same time by Kac [4] and by Ademollo et al. [1]. On the mathematical side Kac and van de Leuer [5], Cheng and Kac [3] have classified all possible superconformal algebras and Kac recently has proved that their classification is complete.

The Neveu-Schwarz, the Ramond and the Topological $N = 2$ superconformal algebras are connected to each other by the spectral flows and/or the topological twists. Therefore, we only consider the *Ramond* $N = 2$ *superconformal algebra*, which is a $\mathbb{Z}_2$-graded space: $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$, with

$$\mathcal{L}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{L_i, H_j, c \mid i, j \in \mathbb{Z}\}, \quad \mathcal{L}_{\bar{1}} = \mathrm{span}_{\mathbb{C}}\{G_i^-, G_j^+ \mid i, j \in \mathbb{Z}\},$$

such that c is a central element and the following relations hold:

$$\begin{aligned}
&[L_i, L_j] = (i-j)L_{i+j} + \tfrac{1}{12}(i^3 - i)\delta_{i+j,0}c,\\
&[L_i, H_j] = -jH_{i+j}, \qquad [H_i, H_j] = \tfrac{1}{3}i\delta_{i+j,0}c,\\
&[L_i, G_j^{\pm}] = (\tfrac{i}{2} - j)G_{i+j}^{\pm}, \qquad [H_i, G_j^{\pm}] = \pm G_{i+j}^{\pm},\\
&[G_i^+, G_j^+] = [G_i^-, G_j^-] = 0, \ [G_i^-, G_j^+] = 2L_{i+j} - (i-j)H_{i+j} + \tfrac{1}{3}(i^2 - \tfrac{1}{4})\delta_{i+j,0}c.
\end{aligned} \tag{1.1}$$

Obviously, the Cartan subalgebra of $\mathcal{L}$ is $\mathcal{H} = \mathbb{C}L_0 + \mathbb{C}H_0 + \mathbb{C}c$, and $Vir = \mathrm{span}_{\mathbb{C}}\{L_m, c \mid m \in \mathbb{Z}\}$ is a Virasoro subalgebra of $\mathcal{L}$, which can be described as the

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universal central extensions of the Lie algebras of differential operators (see [2] for details).

An $\mathcal{L}$-module V is called a *Harish-Chandra module* if V is a direct sum of its finite dimensional weight spaces $V^\lambda = \{v \in V \mid x \cdot v = \lambda(x)v, x \in \mathcal{H}\}$ for all $\lambda \in \mathcal{H}^*$ (the dual of $\mathcal{H}$). Similar to the case of Virasoro algebra, we can define the module of the intermediate series over $\mathcal{L}$:

Definition 1.1 A *module of the intermediate series* over $\mathcal{L}$ is an indecomposable Harish-Chandra module V such that dim $V^\lambda \leqslant 1$ for all $\lambda \in \mathcal{H}^*$.

In this paper, we will consider some properties of $\mathcal{L}$, basing on representations of the above type over the Virasoro algebra.

The paper is arranged as follows. In Section 2, we first consider all possible super-extensions of the Heisenberg-Virasoro type Lie algebra. Our main result in this section is Theorem 2.1. As a conclusion, we obtain that the Ramond $N = 2$ superconformal algebra is a special case of such super-extension. Then we study the modules of the intermediate series in the last section.

Our main result is

Theorem 1.2. Any indecomposable module of the intermediate series V over the Ramond $N = 2$ superconformal algebra is one of modules $RA_{a,b}$, RA_α, RA^β, $RB_{a,b}$, RB_α, RB^β, or one of their quotients for $a, b, \alpha, \beta \in \mathbb{C}$, where $RA_{a,b}$ is defined in (3.25), $RB_{a,b}$ is defined in (3.26), RA_α is defined in (3.39), RA^β is defined in (3.40), (3.41), RB_α is defined in (3.42), RB^β is defined in (3.43).

2. The structure of the Ramond $N = 2$ superconformal algebras

Let $\tilde{\mathcal{L}}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{L_m, H_n \mid m, n \in \mathbb{Z}\}$ be a Heisenberg-Virasoro type algebra (only with 1-dimensional center $\mathbb{C}H_0$) with the following Lie brackets:

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [H_m, H_n] = 0, \quad [L_m, H_n] = -nH_{m+n}. \tag{2.1}$$

Let us consider all possible super-extensions of the Lie algebra $\tilde{\mathcal{L}}_{\bar{0}}$. First assume that $\tilde{\mathcal{L}}_{\bar{1}}$ is an $\tilde{\mathcal{L}}_{\bar{0}}$-module of intermediate series with basis $\{G_i \mid i \in \mathbb{Z}\}$ such that $\tilde{\mathcal{L}}_{\bar{0}} \oplus \tilde{\mathcal{L}}_{\bar{1}}$ is a Lie superalgebra. Then following [8, Theorem 3.2], we can suppose

$$[L_m, G_i] = (a - i + mb)G_{m+i}, \quad [H_n, G_i] = fG_{m+i},$$

for some $a, b, f \in \mathbb{C}, f \neq 0$. Set $[G_i, G_j] = a_{ij}L_{i+j} + b_{ij}H_{i+j}$, for some $a_{ij}, b_{ij} \in \mathbb{C}$. Then from the equation

$$[H_k, [G_i, G_j]] = [[H_k, G_i], G_j] + [G_i, [H_k, G_j]],$$

by letting $k = 0$, we deduce $2f[G_i, G_j] = 0$. Therefore, $[G_i, G_j] = 0$ for all $i, j \in \mathbb{Z}$. This is a trivial extension and not the thing we are interested in. Hence we suppose that $\tilde{\mathcal{L}}_{\bar{1}}$ is a direct sum of two $\tilde{\mathcal{L}}_{\bar{0}}$-modules of intermediate series, with basis $\{G_i^\pm \mid i \in \mathbb{Z}\}$. Then we have the following equations:

$$[L_m, G_i^+] = (a^+ - i + mb^+)G_{m+i}^+, \quad [H_n, G_i^+] = f_1 G_{m+i}^+,$$

$$[L_m, G_i^-] = (a^- - i + mb^-)G_{m+i}^-, \quad [H_n, G_i^-] = f_2 G_{m+i}^-,$$

where $a^+, a^-, b^+, b^-, f_1, f_2 \in \mathbb{C}$, and $(f_1, f_2) \neq (0,0)$. In order to get a nontrivial super-extension, we must have $a^+ = a^-$, denoted by a. Set

$$[G_i^-, G_j^+] = a_{ij} L_{i+j} + b_{ij} H_{i+j}.$$

Since

$$0 = [H_0, [G_i^-, G_j^+]] = (f_1 + f_2)[G_i^-, G_j^+],$$

we can suppose that $f_1 = 1, f_2 = -1$ (replacing H_m by $f_1^{-1} H_m$). Following

$$[L_k, [G_i^-, G_j^+]] = [[L_k, G_i^-], G_j^+] + [G_i^-, [L_k, G_j^+]], \tag{2.2}$$

and setting $k = 0$ in (2.2), we can get that $a = 0$. Following (2.2), we have

$$\begin{aligned} (k - i - j)a_{i,j} &= (-i + kb^-)a_{i+k,j} + (-j + kb^+)a_{i,k+j}, & (2.3)\\ (-i - j)b_{i,j} &= (-i + kb^-)b_{i+k,j} + (-j + kb^+)b_{i,k+j}. & (2.4) \end{aligned}$$

By

$$[G_k^-, [G_i^-, G_j^+]] + [G_i^-, [G_k^-, G_j^+]] = 0, \tag{2.5}$$

we obtain $b_{ij} + b_{kj} = a_{ij}(-k + (i+j)b^-) + a_{kj}(-i + (k+j)b^-)$. Setting $k = i$ gives

$$b_{ij} = a_{ij}(-i + (i+j)b^-). \tag{2.6}$$

Replacing G_k^-, G_i^-, G_j^+ respectively by G_j^+, G_j^+, G_i^- in (2.5) gives

$$b_{ij} = -a_{ij}(-j + (i+j)b^+).$$

Comparing it with (2.6), we can get that

$$b^+ + b^- = 1.$$

Following $[H_k, [G_i^-, G_j^+]] = [[H_k, G_i^-], G_j^+] + [G_i^-, [H_k, G_j^+]]$, we have that

$$a_{k+i,j} = a_{i,k+j}, \quad ka_{i,j} = b_{i,k+j} - b_{k+i,j} \quad \text{for all } \ i, j, k \in \mathbb{Z}.$$

Letting $i = 0, k = i$ gives

$$a_{i,j} = a_{0,i+j} \ \text{ for all } i, j \in \mathbb{Z}. \tag{2.7}$$

Taking $i = 0$ in (2.3), and by (2.7), we obtain that

$$(k - j)a_{0,j} = kb^- a_{k,j} + (-j + kb^+)a_{0,k+j} = (k - j)a_{0,k+j}. \tag{2.8}$$

Let $a_{0,0} = d \in \mathbb{C}$, then by (2.7) and (2.8), we have that $a_{ij} = d$ for all $i, j \in \mathbb{Z}$. Since $b^+ + b^- = 1$, we set $b^- = b$, then

$$b^+ = 1 - b, \quad \text{and} \quad b_{ij} = d(-i + (i+j)b).$$

We can set $d=1$ (replace $G_i^{\pm}$ by $\frac{1}{\sqrt{d}}G_i^{\pm}$), then $\tilde{\mathcal{L}}=\tilde{\mathcal{L}}_{\bar{0}}\oplus\tilde{\mathcal{L}}_{\bar{1}}$ is a superalgebra with (2.1) and the following Lie brackets:

$$\begin{aligned}
&[L_m,G_n^+]=(-n+m(1-b))G_{m+n}^+, &&[L_m,G_n^-]=(-n+mb)G_{m+n}^-,\\
&[H_m,G_n^{\pm}]=\pm G_{m+n}^{\pm}, &&[G_m^{\pm},G_n^{\pm}]=0,\\
&[G_m^-,G_n^+]=L_{m+n}+(-m+(m+n)b)H_{m+n},
\end{aligned}\tag{2.9}$$

where $m,n\in\mathbb{Z}$. Obviously, $\tilde{\mathcal{L}}$ is $\mathbb{Z}$-graded:

$$\tilde{\mathcal{L}}=\underset{n\in\mathbb{Z}}{\oplus}\tilde{\mathcal{L}}_n,\qquad \tilde{\mathcal{L}}_n=\{x\in\tilde{\mathcal{L}}\mid[L_0,x]=nx\}=\mathrm{span}_{\mathbb{C}}\{L_{-n},H_{-n},G_{-n}^{\pm}\}.$$

Now we consider the central extension of $\tilde{\mathcal{L}}$. Suppose $\varphi:\tilde{\mathcal{L}}\times\tilde{\mathcal{L}}\to\mathbb{C}$ is a 2-cocycle of $\tilde{\mathcal{L}}$, we define a linear map $f:\tilde{\mathcal{L}}\to\mathbb{C}$ as follows:

$$\begin{aligned}
&f(\tilde{\mathcal{L}}_i)=\tfrac{1}{i}\varphi(L_0,\tilde{\mathcal{L}}_i),\quad i\neq 0, &&f(L_0)=\tfrac{1}{2}\varphi(L_1,L_{-1}),\\
&f(H_0)=\varphi(L_1,H_{-1}), &&f(G_0^{\pm})=\pm\varphi(H_1,G_{-1}^{\pm}).
\end{aligned}\tag{2.10}$$

If we define another 2-cocycle of $\tilde{\mathcal{L}}$, $\psi:\tilde{\mathcal{L}}\times\tilde{\mathcal{L}}\to\mathbb{C}$ satisfying $\psi=\varphi-\varphi_f$, where $\varphi_f(x,y)=f([x,y])$, then

$$\begin{aligned}
\psi(L_0,\tilde{\mathcal{L}}_i)&=\varphi(L_0,\tilde{\mathcal{L}}_i)-\varphi_f(L_0,\tilde{\mathcal{L}}_i)=0,\quad i\neq 0,\\
2\psi(L_0,L_0)&=\psi([L_1,L_{-1}],L_0)=0,
\end{aligned}$$

Similarly, we have that $\psi(L_0,H_0)=\psi(L_0,G_0^{\pm})=0$. Thus

$$\psi(L_0,\tilde{\mathcal{L}}_i)=0\quad\text{for all } i\in\mathbb{Z}.\tag{2.11}$$

Furthermore,

$$i\psi(\tilde{\mathcal{L}}_i,\tilde{\mathcal{L}}_j)=\psi([L_0,\tilde{\mathcal{L}}_i],\tilde{\mathcal{L}}_j)=\psi(L_0,[\tilde{\mathcal{L}}_i,\tilde{\mathcal{L}}_j])-\psi(\tilde{\mathcal{L}}_i,[L_0,\tilde{\mathcal{L}}_j])=-j\psi(\tilde{\mathcal{L}}_i,\tilde{\mathcal{L}}_j),$$

therefore,

$$\psi(\tilde{\mathcal{L}}_i,\tilde{\mathcal{L}}_j)=0\quad\text{if}\ \ i+j\neq 0.\tag{2.12}$$

Now let us consider $\psi(L_i,L_{-i})$. It follows (2.10) that

$$\psi(L_1,L_{-1})=\varphi(L_1,L_{-1})-\varphi_f(L_1,L_{-1})=0.$$

Then

$$(i-2)\psi(L_i,L_{-i})=\psi([L_{i-1},L_1],L_{-i})=(i+1)\psi(L_{i-1},L_{-i+1}).$$

Set $\psi(L_i,L_{-i})=l_i$, we obtain $l_i=\frac{i+1}{i-2}l_{i-1}$ for $i\neq 2$, i.e.,

$$l_i=\tfrac{i^3-i}{6}l_2\quad\text{for all}\quad i\geqslant 3.$$

Then we can rewrite $\psi(L_i, L_{-i})$ as follows

$$\psi(L_i, L_{-i}) = \tfrac{i^3-i}{6}c_L \quad \text{for all } i \in \mathbb{Z}, \tag{2.13}$$

where $c_L \in \mathbb{C}$. Similar to the argument about $\psi(L_i, L_{-i})$, we can obtain that

$$\psi(H_i, H_{-i}) = ic_H, \qquad \psi(L_i, H_{-i}) = \tfrac{i(i-1)}{2}c_{HL}, \tag{2.14}$$

$$\psi(H_i, G^{\pm}_{-i}) = \psi(L_i, G^{\pm}_{-i}) = \psi(G^{\pm}_i, G^{\pm}_{-i}) = 0 \quad \text{for all } i \in \mathbb{Z}, \tag{2.15}$$

where $c_H, c_{HL} \in \mathbb{C}$. Finally, we consider $\psi(G^-_i, G^+_{-i})$. Set $\psi(G^-_0, G^+_0) = c_G$, by (2.9) and (2.14), we have

$$\begin{aligned}\psi(G^-_1, G^+_{-1}) &= \psi([G^-_0, H_1], G^+_{-1})\\ &= \psi(G^-_0, [H_1, G^+_{-1}]) - \psi(H_1, [G^-_0, G^+_{-1}]) = c_G + c_{HL} + bc_H,\end{aligned}$$

and $\psi(G^-_1, G^+_{-1}) = \psi(G^-_1, [H_{-1}, G^+_0]) = c_G + (1-b)c_H$. Hence $c_G + c_{HL} + bc_H = c_G + (1-b)c_H$, i.e.,

$$c_{HL} = (1-2b)c_H. \tag{2.16}$$

Then

$$\begin{aligned}\psi(G^-_i, G^+_{-i}) &= \psi([G^-_0, H_i], G^+_{-i})\\ &= \psi(G^-_0, [H_i, G^+_{-i}]) - \psi(H_i, [G^-_0, G^+_{-i}]) = c_G + \big(\tfrac{i(i+1)}{2} - ib\big)c_H.\end{aligned} \tag{2.17}$$

Note that

$$[L_i, [G^-_j, G^+_k]] = [[L_i, G^-_j], G^+_k] + [G^-_j, [L_i, G^+_k]].$$

If we suppose $i + j + k = 0$, then

$$\begin{aligned}&\tfrac{i^3-i}{6}c_L + (-j + (k+j)b)\tfrac{i(i-1)}{2}c_{HL}\\ &= (-j + ib)\big(c_G + (\tfrac{-k(1-k)}{2} + kb)c_H\big) + (-k + i(1-b))\big(c_G + (\tfrac{j(j+1)}{2} - jb)c_H\big).\end{aligned} \tag{2.18}$$

By (2.16), and setting $j = 0$ in (2.18), we have that

$$\tfrac{i^2-1}{6}c_L - 2c_G = i^2(b - b^2)c_H. \tag{2.19}$$

Letting $i = 1$ in (2.19), we can obtain that

$$c_G = \tfrac{b^2-b}{2}c_H, \qquad c_L = 6(b - b^2)c_H. \tag{2.20}$$

Then we have the following theorem:

Theorem 2.1. The possible nontrivial super-extensions of the Heisenberg-Virasoro type algebra (2.1) are the following superalgebras:

$$\hat{\mathcal{L}} = \mathrm{span}_{\mathbb{C}}\{L_i, H_j, G^{\pm}_k, c_H \mid c_H \in \mathbb{C}, i, j, k \in \mathbb{Z}\},$$

where c_H is a central element and the following relations hold:

$$
\begin{aligned}
&[L_i, L_j] = (i-j)L_{i+j} + (i^3 - i)(b - b^2)c_H\delta_{i+j,0}, \\
&[L_i, H_j] = -jH_{i+j} + \tfrac{i(i-1)}{2}(1-2b)c_H\delta_{i+j,0}, \quad [H_i, H_j] = ic_H\delta_{i+j,0}, \\
&[L_i, G_j^+] = (-j + i(1-b))G_{i+j}^+, \qquad [L_i, G_j^-] = (-j + ib)G_{i+j}^-, \\
&[H_i, G_j^\pm] = \pm G_{i+j}^\pm, \qquad [G_i^+, G_j^+] = [G_i^-, G_j^-] = 0, \\
&[G_i^-, G_j^+] = L_{i+j} + (-i + (i+j)b)H_{i+j} + \tfrac{i(i+1-2b)+b^2-b}{2}c_H\delta_{i+j,0}.
\end{aligned}
\tag{2.21}
$$

If $b = \frac{1}{2}$, then $\hat{\mathcal{L}} = \mathcal{L}$. That is to say, Ramond $N = 2$ superconformal algebra $\mathcal{L}$ is a special case of $\hat{\mathcal{L}}$.

3. The modules of intermediate series over $\mathcal{L}$

§3.0 Let $V = V_0 \oplus V_1$ be any indecomposable $\mathcal{L}$-module with $\dim V_\alpha^\lambda \leqslant 1$ for all $\lambda \in \mathcal{H}^*, \alpha \in \mathbb{Z}/2\mathbb{Z}$, where

$$V_\alpha^\lambda = \{v \in V_\alpha \mid L_0 \cdot v = \lambda(L_0)v, H_0 \cdot v = \lambda(H_0)v\}.$$

We also have the following:

$$V = \Big(\underset{k\in\mathbb{Z}}{\oplus} V_0^{a+k}\Big) \oplus \Big(\underset{k\in\mathbb{Z}}{\oplus} V_1^{a+k}\Big). \tag{3.1}$$

One sees that c acts trivially on V (see, e.g., [6, 9]). So we can omit c in (1.1).

Now we consider all possibilities of V_0 and V_1 case by case below. Let us recall the definition of Vir-modules $A_{a,b}, A(\alpha), B(\beta)$ (see [6]). They all have a basis $\{x_i \mid i \in \mathbb{Z}\}$ such that for $i, j \in \mathbb{Z}$,

$$
\begin{aligned}
&A_{a,b}: \quad L_i x_j = (a - j + ib)x_{i+j}. \\
&A(\alpha): \quad L_i x_j = -(i+j)x_{i+j}, \ j \neq 0, \quad L_i x_0 = -i(1 + (i+1)\alpha)x_i. \\
&B(\beta): \quad L_i x_j = -jx_{i+j}, \ i+j \neq 0, \quad L_i x_{-i} = i(1 + (i+1)\beta)x_0.
\end{aligned}
\tag{3.2}
$$

§3.1 Suppose both of V_0, V_1 have the form $A_{a,b}, a, b \in \mathbb{C}$. Then we choose a basis $\{x_i \mid i \in \mathbb{Z}\}$ of V_0 and a basis $\{y_j \mid j \in \mathbb{Z}\}$ of V_1 such that

$$L_i x_j = (a - j + ib)x_{i+j}, \quad L_i y_j = (a' - j + ib')y_{i+j}, \tag{3.3}$$

$$H_i x_j = f_{ij}x_{i+j}, \quad H_i y_j = f'_{ij}y_{i+j}, \tag{3.4}$$

$$G_i^\pm x_j = a_{ij}^\pm y_{i+j}, \quad G_i^\pm y_j = b_{ij}^\pm x_{i+j}, \tag{3.5}$$

where $a, a', b, b', f_{ij}, f'_{ij}, a_{ij}^\pm, b_{ij}^\pm \in \mathbb{C}$. We have $a = a'$ by applying L_0 to the first formula of (3.5). By (1.1), we have

$$(\tfrac{i}{2} - j)a_{i+j,k}^\pm = (a - (k+j) + ib')a_{j,k}^\pm - (a - k + ib)a_{j,i+k}^\pm, \tag{3.6}$$

and $2(a-k+(i+j)b)-(i-j)f_{i+j,k}=a^+_{j,k}b^-_{i,k+j}+a^-_{i,k}b^+_{j,k+i}$. Let $i=j$, we get

$$a^+_{i,k}b^-_{i,k+i}+a^-_{i,k}b^+_{i,k+i}=2(a-k+2ib). \tag{3.7}$$

From (3.7), we know that for all fixed $i,k\in\mathbb{Z}$,

$$a^+_{i,j}=a^-_{i,j}=0,\ b^+_{k,l}=b^-_{k,l}=0 \text{ only for finitely many } j \text{ and finitely many } l. \tag{3.8}$$

Applying $[G^\pm_i,G^\pm_j]=0$ to x_k gives $a^\pm_{i,j}b^\pm_{i,k+j}+a^\pm_{i,k}b^\pm_{j,k+i}=0$. Letting $i=j$ gives

$$a^\pm_{i,k}b^\pm_{i,k+i}=0. \tag{3.9}$$

Therefore, by (3.7) and (3.9), for any $i,k\in\mathbb{Z}$ with $a-k+2ib\neq 0$, we have

$$a^+_{ik}a^-_{ik}=0,\quad (a^+_{ik})^2+(a^-_{ik})^2\neq 0,\quad \text{(similar relations for } b^\pm_{ik}). \tag{3.10}$$

For convenience, we omit the superscript "$\pm$" in $a^\pm_{ij}$. Let $i=2j$ and $i=-2j$ in (3.6) respectively, we get

$$(a-k+2jb)a_{j,k+2j}=(a-(k+j)+2jb')a_{j,k}, \tag{3.11}$$

$$-2ja_{-j,k}=(a-(k+j)-2jb')a_{j,k}-(a-k-2jb)a_{j,k-2j}. \tag{3.12}$$

Multiplying (3.12) by $a-(k-j)+2jb'$ and replacing the last term by (3.11), we get

$$\begin{aligned}&-2j(a-(k-j)+2jb')a_{-j,k}\\&=(a-(k-j)+2jb')(a-(k+j)-2jb')a_{j,k}-(a-k-2jb)(a-(k-2j)+2jb)a_{j,k}\\&=2j(a-(k-j)+2jb'+2jt)a_{j,k},\end{aligned}$$

where $t=b'^2-(b+\frac{1}{2})^2$. Similarly, let $j=-j$, $i=2j$ and $j=-j$, $i=-2j$ in (3.6), we can obtain that

$$2j(a-(k+j)-2jb')a_{j,k}=-2j(a-(k+j)-2jb'-2jt)a_{-j,k}.$$

It follows that

$$\begin{aligned}&\big((a-(k+j)-2jb')(a-(k-j)+2jb')\\&-((a-k-j-2jb')-2jt)((a-k+j+2jb')+2jt)\big)a_{jk}=0,\end{aligned}$$

which gives

$$4j^2t(t+2b'+1)a_{jk}=0. \tag{3.13}$$

By (3.8), there at least exists one k_0 such that $a^+_{1,k_0}\neq 0$, or $a^-_{1,k_0}\neq 0$. (If $a^\pm_{1,k}=0$ for all k, by letting $j=1$ in (3.6) we get $a^\pm_{ik}=0$ for $i,k\in\mathbb{Z}, i\neq 2$. By letting $i=j=1$ in (3.6), we get $a^\pm_{2,k}=0$.) Thus it follows from (3.13) that

$$b'=\pm(b+\tfrac{1}{2}),\quad \text{or}\quad b'=-1\pm(b+\tfrac{1}{2}).$$

Case 1. $b' = b + \frac{1}{2}$.

First suppose $a - k + 2b \neq 0$ for all $k \in \mathbb{Z}$. Letting $j = 1$ in (3.11), we obtain (again we omit the superscript "$\pm$" in x_0, x_1 for the time being)

$$a_{1,k} = \begin{cases} x_0, & k \text{ is even}, \\ x_1, & k \text{ is odd}. \end{cases}$$

Let $j = 1$, and let i, k be odd in (3.6), then

$$\begin{aligned}(\tfrac{i}{2} - 1)a_{i+1,k} &= (a - (k+1) + ib')a_{1,k} - (a - k + ib)a_{1,k+i} \\ &= (a - (k+1) + ib')x_1 - (a - k + ib)x_0.\end{aligned}$$

By (3.11) and $b' = b + \frac{1}{2}$, we also have

$$\begin{aligned}(\tfrac{i}{2} - 1)a_{i+1,k} &= (\tfrac{i}{2} - 1)a_{i+1,k+2(i+1)} \\ &= (a - (k + 2(i+1) + 1) + ib')x_1 - (a - (k + 2(i+1)) + ib)x_0.\end{aligned}$$

Obviously, we get $x_0 = x_1$. Similar to the arguments after (3.13), we have

$$a_{ij}^{\pm} = d_1^{\pm} \quad \text{for all} \ \ i, j \in \mathbb{Z}, \tag{3.14}$$

where $d_1^{\pm}$ are constants, and by (3.10),

$$d_1^+ d_1^- = 0, \quad (d_1^+)^2 + (d_1^-)^2 \neq 0. \tag{3.15}$$

Now we suppose that $a - k' + 2b = 0$ for some $k' \in \mathbb{Z}$. It follows from (3.11) that $(a - k + 2b)a_{1,k} = (a - k + 2b)a_{1,k+2}$. Then

$$a_{1k} = \begin{cases} x_0, & k > k', \ k \text{ is even}, \\ x_1, & k > k', \ k \text{ is odd}. \end{cases} \qquad a_{1k} = \begin{cases} y_0, & k \leqslant k', \ k \text{ is even}, \\ y_1, & k \leqslant k', \ k \text{ is odd}. \end{cases}$$

By (3.6), we get

$$a_{ik} = \begin{cases} x_0, & k > k', \ k + i - 1 > k', \ \text{and } k, \ i - 1 \text{ are even}, \\ x_1, & k > k', \ k + i - 1 > k', \ \text{and } k, \ i - 1 \text{ are odd}, \end{cases}$$

and

$$a_{ik} = \begin{cases} y_0, & k \leqslant k', \ k + i - 1 \leqslant k', \ \text{and } k, \ i - 1 \text{ are even}, \\ y_1, & k \leqslant k', \ k + i - 1 \leqslant k', \ \text{and } k, \ i - 1 \text{ are odd}. \end{cases}$$

Now choose some $k, j \in \mathbb{Z}$, such that $a - k + 2jb \neq 0$, $k \leqslant k'$, $k + j - 1 \leqslant k'$, $k + 2j > k'$, $k + 3j - 1 > k'$, and one of k and j is even, and another is odd. Then by (3.11), we have

$$(a - k + 2jb)a_{j,k} = (a - k + 2jb)a_{j,k+2j}.$$

Therefore,

$$x_0 = y_0, \quad x_1 = y_1.$$

Similar to the argument above, we again have (3.14) and (3.15).

Case 2. $b' = -(b + \frac{1}{2})$.

Let $i = 2j$ and $i = -2j$ respectively in (3.6), we have

$$(a - k - 2j - 2jb)a_{j,k} = (a - k + 2jb)a_{j,k+2j}, \tag{3.16}$$

$$-2ja_{-j,k} = (a - k + 2jb)a_{j,k} - (a - k - 2jb)a_{j,k-2j} = -2ja_{j,k}, \tag{3.17}$$

where the second equality of (3.17) follows from (3.16) by replaced k by $k - 2j$. Hence, $a_{j,k} = a_{-j,k}$ for all $k, j \in \mathbb{Z}$. Using it, again by (3.6), we deduce that

$$\begin{aligned}&(a - (k + j) + ib')a_{j,k} - (a - k + ib)a_{j,k+i} = (\tfrac{i}{2} - j)a_{i+j,k}\\ &= (\tfrac{i}{2} - j)a_{-i-j,k} = -(a - (k - j) - ib')a_{-j,k} - (a - k - ib)a_{-j,k-i}.\end{aligned}$$

Hence

$$(a - k - ib)a_{j,k-i} - 2(a - k)a_{j,k} + (a - k + ib)a_{j,k+i} = 0. \tag{3.18}$$

Let $j = 1$ in (3.16) and then replace k by $k + 2$ in the new equality, we can obtain

$$a_{1,k+4} = \tfrac{(a-k-4-2b)(a-k-2-2b)}{(a-k-2+2b)(a-k+2b)}a_{1,k}.$$

Similarly, we can have a formula for $a_{1,k-4}$. Then let $i = 4$ in (3.18), we have

$$\begin{aligned}&\big((a - k - 4b)\tfrac{(a-k+4+2b)(a-k+2+2b)}{(a-k+2-2b)(a-k-2b)} - 2(a - k)\\ &+(a - k + 4b)\tfrac{(a-k-4-2b)(a-k-2-2b)}{(a-k-2+2b)(a-k+2b)}\big)a_{1,k} = 0.\end{aligned}$$

By (3.11) and the discussion after (3.13), we know the coefficient of $a_{1,k}$ must be zero. We obtain $b = -1$ or $-\frac{1}{2}$. Note that the case of $b = -\frac{1}{2}, b' = 0$ is contained in Case 1. So we only need to consider the case of $b = -1, b' = \frac{1}{2}$.

Let $j = 1$ in (3.11), then $(a - k)a_{1,k}$ is a constant for all even k or all odd k. We suppose that

$$(a - k)a_{1,k} = \begin{cases} x_0, & k \text{ is even,} \\ x_1, & k \text{ is odd.}\end{cases}$$

Let $i = j = 1$ in (3.18), we can obtain that $x_0 = x_1$. That is to say, $(a - k)a_{1,k}$ is a constant for all $k \in \mathbb{Z}$. If $a - k_1 = 0$ for some $k_1 \in \mathbb{Z}$, then $a_{1,k} = \frac{a-k_1}{a-k}a_{1,k_1} = 0$ for all $k \neq k_1$, a contradiction with (3.8). Thus $a - k \neq 0$ for all $k \in \mathbb{Z}$, i.e., $a \notin \mathbb{Z}$. Denote $(a - k)a^{\pm}_{1,k}$ by $d^{\pm}_2$, where $d^{\pm}_2 \in \mathbb{C}$ are constants. By (3.6), let $j = 1$, we get

$$(\tfrac{i}{2} - 1)a_{i+1,k} = (a - (k + 1) + \tfrac{i}{2})a_{1,k} - (a - k - 1)a_{1,k+1},$$

then $a_{i,k} = a_{1,k}$ for all $i \in \mathbb{Z}, i \neq 3$. Again by (3.6), let $j = 2$, $i = 1$, we have

$$-\tfrac{3}{2}a_{3,k} = (a-(k+2)+\tfrac{1}{2})a_{2,k} - (a-k-1)a_{2,k+1},$$

then $a_{3,k} = a_{1,k}$. Therefore, $a^{\pm}_{ij} = (a-j)^{-1}d^{\pm}_2$ for all $i, j \in \mathbb{Z}$, and by (3.10),

$$d^+_2 d^-_2 = 0, \quad (d^+_2)^2 + (d^-_2)^2 \neq 0.$$

Case 3. $b' = -b - \frac{3}{2}$.

Following the arguments in Case 2, we have $2ja_{j,k} = 2ja_{-j,k-2j}$, and

$$(a-k-j+ib')a_{j,k} + (a-k-j-2i-ib')a_{j,k+2i} = 2(a-k-i-j)a_{j,k+i}, \tag{3.19}$$

$$\begin{aligned}\Big((a-k-1+4b')\tfrac{(a-k-3-2b')(a-k-5-2b')}{(a-k-1+2b')(a-k-3+2b')} - 2(a-k-5)\\ +(a-k-9-4b')\tfrac{(a-k-7+2b')(a-k-5+2b')}{(a-k-9-2b')(a-k-7-2b')}\Big)a_{1,k+4} = 0.\end{aligned} \tag{3.20}$$

Then we obtain that $b = -\frac{3}{2}$ or -1. The case of $b = -1$ $(b' = -\frac{1}{2})$ is contained in Case 1. So we consider the case of $b = -\frac{3}{2}, b' = 0$. By (3.11), let $j = 1$, we have

$$(a-k-1)a_{1,k} = (a-k-3)a_{1,k+2},$$

then $(a-k-1)a_{1,k}$ is a constant for all even k or all odd k. By (3.19), similar to the argument in Case 2, we can obtain that $(a-k-1)a_{1,k}$ is a constant for all $k \in \mathbb{Z}$. If $a - k_1 - 1 = 0$ for some k_1, then $a_{1,k} = \frac{a-k_1-1}{a-k-1}a_{1,k_1} = 0$ for all $k \neq k_1$. Also a contradiction with (3.8). Therefore, $a \notin \mathbb{Z}$. Now we denote $(a-k-1)a_{1,k}$ by d_3, i.e. $a_{1,k} = (a-k-1)^{-1}d_3$, for all $k \in \mathbb{Z}$. Let $j = 1$ in (3.6), then we have that

$$(\tfrac{i}{2}-1)a_{i+1,k} = (a-k-1)a_{1,k} - (a-k-\tfrac{3}{2}i)a_{1,k+i}.$$

Then

$$a_{i,k} = a_{1,k+i-1} = (a-k-i)^{-1}d_3 \ \text{ for } i \neq 3.$$

Let $i = 1, j = 2$ in (3.6), we get $-\frac{3}{2}a_{3,k} = (a-k-2)a_{2,k} - (a-k-\frac{3}{2})a_{2,k+1} = -\frac{3}{2}a_{2,k+1}$, then $a_{3,k} = (a-k-3)^{-1}d_3$. Therefore, $a^{\pm}_{ij} = (a-i-j)^{-1}d^{\pm}_3$ for all $i, j \in \mathbb{Z}$, and by (3.10),

$$d^+_3 d^-_3 = 0, \quad (d^+_3)^2 + (d^-_3)^2 \neq 0.$$

Case 4. $b' = b - \frac{1}{2}$.

Note that if we act $(\frac{i}{2}-j)G^{\pm}_{i+j} = [L_i, G^{\pm}_j]$ on y_k, we can obtain that

$$(\tfrac{i}{2}-j)b^{\pm}_{i+j,k} = (a-(k+j)+ib)b^{\pm}_{j,k} - (a-k+ib')b^{\pm}_{j,i+k}. \tag{3.21}$$

Similar to the discussion in case 1 $(b = b' + \frac{1}{2})$, we have $b^{\pm}_{ij} = d'^{\pm}_1$, $i, j \in \mathbb{Z}$ for some $d'^{\pm}_1 \in \mathbb{C}$, and by (3.10),

$$d'^+_1 d'^-_1 = 0, \quad (d'^+_1)^2 + (d'^-_1)^2 \neq 0.$$

Then it follows from (3.7) that

$$a_{ij}^{\pm} = 2(a-k+2ib)(d_1'^{\mp})^{-1}.$$

Until now, we get that

$$a_{ij}^{\pm} = \begin{cases} d_1^{\pm}, & b' = b + \frac{1}{2}, \\ (a-j)^{-1}d_2^{\pm}, & b' = \frac{1}{2}, \ \ b = -1, \\ (a-i-j)^{-1}d_3^{\pm}, & b' = 0, \ \ b = -\frac{3}{2}, \\ 2(a-j+2ib)(d_1'^{\mp})^{-1}, & b' = b - \frac{1}{2}. \end{cases}$$

Again using (3.17), following the same arguments about $a_{i,j}^{\pm}$, we have that

$$b_{ij}^{\pm} = \begin{cases} d_1'^{\pm}, & b = b' + \frac{1}{2}, \\ (a-j)^{-1}d_2'^{\pm}, & b = \frac{1}{2}, \ \ b' = -1, \\ (a-i-j)^{-1}d_3'^{\pm}, & b = 0, \ \ b' = -\frac{3}{2}, \\ 2(a-j+2i(b+\frac{1}{2}))(d_1^{\mp})^{-1}, & b = b' - \frac{1}{2}. \end{cases}$$

Obviously only the following two cases can occur:

$$\begin{array}{lll} a_{ij}^{\pm} = d^{\pm}, & b_{ij}^{\pm} = 2(a-j+2i(b+\frac{1}{2}))(d^{\mp})^{-1}, & b' = b + \frac{1}{2}. \\ b_{ij}^{\pm} = d'^{\pm}, & a_{ij}^{\pm} = 2(a-j+2ib)(d'^{\mp})^{-1}, & b = b' + \frac{1}{2}, \end{array} \tag{3.22}$$

for some $d^{\pm}, d'^{\pm} \in \mathbb{C}$. Together with (3.10), we obtain that

$$\begin{array}{ll} (d^{+})^2 + (d^{-})^2 = 0, & d^{+}d^{-} = 0, \ \ \text{and} \\ (d'^{+})^2 + (d'^{-})^2 = 0, & d'^{+}d'^{-} = 0. \end{array}$$

By rescaling basis $\{y_i \mid i \in \mathbb{Z}\}$ (or $\{x_i \mid i \in \mathbb{Z}\}$) if necessary, we can suppose $d^{\pm} = 1$ (or $d'^{\pm} = 1$). Then we rewrite (3.22):

$$\begin{array}{llll} a_{ij}^{+} \ (\text{or } a_{ij}^{-}) = 1, & b_{ij}^{-} \ (\text{or } b_{ij}^{+}) = & 2(a-j+2i(b+\frac{1}{2})), & b' = b + \frac{1}{2}. \\ b_{ij}^{+} \ (\text{or } b_{ij}^{-}) = 1, & a_{ij}^{-} \ (\text{or } a_{ij}^{+}) = & 2(a-j+2ib), & b' = b - \frac{1}{2}. \end{array} \tag{3.23}$$

Now we consider one of the cases of (3.23):

$$a_{ij}^{+} = 1, \ b_{ij}^{-} = 2(a-j+2i(b+\tfrac{1}{2})), \ a_{ij}^{-} = b_{ij}^{+} = 0, \ b' = b + \tfrac{1}{2}.$$

We want to determine the action of H_i on V. Set $f_{00} = f, f'_{00} = f'$. Since $[H_0, G_i^{+}] = G_i^{+}$, we have that $f' = f + 1$. Similar to the arguments of [8], we get the following cases.

Case 5. $a - b, a - b' \notin \mathbb{Z}$. We have some cases as follows:

Subcase 5.1. $f_{ij} = f$, $\quad f'_{ij} = f + 1$; $\quad b' = b + \frac{1}{2}$.

Subcase 5.2. $f_{ij} = \frac{a-j}{a-i-j} f$, $\quad f'_{ij} = f + 1$; $\quad b = 0, \ b' = \frac{1}{2}$.

Subcase 5.3. $f_{ij} = \frac{a-i-j}{a-j} f$, $\quad f'_{ij} = f + 1$; $\quad b = -1, \ b' = -\frac{1}{2}$.

Subcase 5.4. $f_{ij} = f$, $\quad f'_{ij} = \frac{a-j}{a-i-j}(f+1)$; $\quad b = -\frac{1}{2}, \ b' = 0$.

Subcase 5.5. $f_{ij} = f$, $\quad f'_{ij} = \frac{a-i-j}{a-j}(f+1)$; $\quad b = -\frac{3}{2}, \ b' = -1$.

Note that

$$[H_i, G_j^+] = G_{i+j}^+. \tag{3.24}$$

Acting it on x_k, we compare the coefficients on the two sides, then we get contradictions for Subcases 5.2–5.5. And in Subcase 5.1, follows $[H_i, G_j^-] \cdot y_k = -G_{i+j}^- \cdot y_k$, one can get that $f = -2b - 2$. Then we obtain a representation $RA_{a,b}$ of $\mathcal{L}$ with basis $\{x_i, y_i \,|\, i \in \mathbb{Z}\}$ and the actions:

$$\begin{aligned} RA_{a,b}\colon L_i x_j &= (a - j + ib)x_{i+j}, \; L_i y_j = (a - j + i(b + \tfrac{1}{2}))y_{i+j}, \\ H_i x_j &= -(2b+2)x_{i+j}, \quad H_i y_j = -(2b+1)y_{i+j}, \\ G_i^- x_j &= G_i^+ y_j = 0, \qquad G_i^+ x_j = y_{i+j}, \quad G_i^- y_j = 2(a + i - j + 2ib)x_{i+j}, \end{aligned} \tag{3.25}$$

(together with $cx_i = cy_i = 0$ for all $i, j \in \mathbb{Z}$).

Obviously:

(i) As Vir-modules, $V_0 \cong A_{a,b}$ and $V_1 \cong A_{a,b'}$, where b and b' have some relations.

(ii) For all $i \in \mathbb{Z}$, H_i acts as constants on V_0 and V_1.

Case 6. $a - b \in \mathbb{Z}$, then $a - b' \notin \mathbb{Z}$. Similar to the discussion in case 5, we have the following subcases:

Subcase 6.1.	$f_{ij} = 0,$	$f'_{ij} = 1;$	$b = 0,$	$b' = \frac{1}{2}.$
Subcase 6.2.	$f_{ij} = 0,$	$f'_{ij} = 1;$	$b = -1,$	$b' = -\frac{1}{2}.$
Subcase 6.3.	$f_{ij} = f \neq 0,$	$f'_{ij} = f + 1;$	$b = -\frac{1}{2},$	$b' = 0.$
Subcase 6.4.	$f_{ij} = f \neq 0,$	$f'_{ij} = \frac{a-i}{a-i-j}(f+1);$	$b = -\frac{1}{2},$	$b' = 0.$
Subcase 6.5.	$f_{ij} = \frac{a-j}{a-i-j} f,$	$f'_{ij} = f + 1;$	$b = 0,$	$b' = \frac{1}{2}.$
Subcase 6.6.	$f_{ij} = \frac{a-i-j}{a-j} f,$	$f'_{ij} = f + 1;$	$b = -1,$	$b' = -\frac{1}{2}.$
Subcase 6.7.	$f_{ij} = f,$	$f'_{ij} = \frac{a-i-j}{a-j}(f+1);$	$b = -\frac{3}{2},$	$b' = -1.$

Again by (3.24), we know that only Subcases 6.2 and 6.3 can occur. It is not difficult to see that they are contained in (3.25).

Case 7. $a - b' \in \mathbb{Z}$, then $a - b \notin \mathbb{Z}$. Similar to the discussion of case 6, we obtain that if $a_{ij}^+ = 1,\; b_{ij}^- = 2(a - j + 2i(b + \frac{1}{2})),\; a_{ij}^- = b_{ij}^+ = 0,\; b' = b + \frac{1}{2}$, the module V has the form of $RA_{a,b}$, for some $a, b \in \mathbb{C}$.

Similarly, we can write the other three cases of (3.23):

$$\begin{aligned} RB_{a,b}\colon L_i x_j &= (a - j + ib)x_{i+j}, L_i y_j = (a - j + i(b + \tfrac{1}{2}))y_{i+j}, \\ H_i x_j &= (2b+2)x_{i+j}, \qquad H_i y_j = (2b+1)y_{i+j}, \\ G_i^+ x_j &= G_i^- y_j = 0, \qquad G_i^- x_j = y_{i+j}, \quad G_i^+ y_j = 2(a + i - j + 2ib)x_{i+j}, \end{aligned} \tag{3.26}$$

when $a_{ij}^- = 1,\; b_{ij}^+ = 2(a - j + 2i(b + \frac{1}{2})),\; a_{ij}^+ = b_{ij}^- = 0,\; b' = b + \frac{1}{2}$.

$$
\begin{aligned}
RA'_{a,b}\colon\ & L_ix_j &&= (a-j+ib)x_{i+j}, && L_iy_j &&= (a-j+i(b-\tfrac{1}{2}))y_{i+j}, \\
& H_ix_j &&= -2b\ x_{i+j}, && H_iy_j &&= -(2b+1)y_{i+j}, \\
& G_i^+x_j &&= G_i^-y_j = 0, && G_i^+y_j &&= \ x_{i+j}, \quad G_i^-x_j = 2(a-j+2ib)y_{i+j},
\end{aligned}
\tag{3.27}
$$

when $a_{ij}^- = 2(a-j+2ib),\ b_{ij}^+ = 1,\ a_{ij}^+ = b_{ij}^- = 0,\ b' = b-\frac{1}{2}$.

$$
\begin{aligned}
RB'_{a,b}\colon\ & L_ix_j &&= (a-j+ib)x_{i+j}, && L_iy_j &&= (a-j+i(b-\tfrac{1}{2}))y_{i+j}, \\
& H_ix_j &&= 2b\ x_{i+j}, && H_iy_j &&= (2b+1)y_{i+j}, \\
& G_i^-x_j &&= G_i^+y_j = 0, && G_i^-y_j &&= \ x_{i+j}, \quad G_i^+x_j = 2(a-j+2ib)y_{i+j},
\end{aligned}
\tag{3.28}
$$

when $a_{ij}^+ = 2(a-j+2ib),\ b_{ij}^- = 1,\ a_{ij}^- = b_{ij}^+ = 0,\ b' = b-\frac{1}{2}$.

It is not difficult to see that $RA_{a,b} \cong RA'_{a,b+\frac{1}{2}}$, $RB_{a,b} \cong RB'_{a,b+\frac{1}{2}}$.

§3.2 Now we consider all the possible deformations of the representations which defined in section 3.1.

Case 1. Suppose V is an indecomposable module which has the same composition factors as those of $RA_{a,b}$ (in this case $RA_{a,b}$ is reducible). Let V' be a non-zero submodule of V.

Subcase 1.1. There exists $x_i \in V'$ for some $i \in \mathbb{Z}$. By the sixth equation of (3.25), we obtain that $y_j \in V'$ for all $j \in \mathbb{Z}$. By the last equation of (3.25),

$$
G_i^-y_{k-i} = 2(a+i-k+i+2ib)x_k = 2(a-k+2(b+1)i)x_k. \tag{3.29}
$$

we see that V' is a proper submodule of V if and only if $a = k_0,\ b = -1$ for some $k_0 \in \mathbb{Z}$. In this case we can suppose $a = 0,\ b = -1$. Then

$$
V' = \mathrm{span}_{\mathbb{C}}\{x_i, y_j \mid i, j \in \mathbb{Z},\ i \neq 0\}
$$

is a nontrivial irreducible submodule of V, with the following relations:

$$
\begin{aligned}
& L_ix_k &&= -(i+k)x_{i+k}, && L_iy_j &&= -(\tfrac{i}{2}+j)y_{i+j}, \\
& H_ix_k &&= 0, && H_iy_j &&= y_{i+j}, \\
& G_i^-x_k &&= G_i^+y_j = 0, && G_i^+x_k &&= y_{i+k}, \quad G_i^-y_j = -2(i+j)x_{i+j},
\end{aligned}
\tag{3.30}
$$

for all $j, k \in \mathbb{Z}$ and $k \neq 0$.

In order to determine all possible actions on V, we suppose that

$$
L_ix_0 = l_ix_i, \quad H_ix_0 = h_ix_i,\ G_i^{\pm}x_0 = g_i^{\pm}y_i \quad \text{for all} \quad i \in \mathbb{Z}.
$$

Act $[L_{i-1}, H_1] = -H_i$ on x_0, we can obtain that

$$
h_i = ih_1 \quad \text{for all} \quad i \in \mathbb{Z}. \tag{3.31}
$$

Applying $[L_i, G_j^-] = (\frac{i}{2}-j)G_{i+j}^-$ to x_0, we obtain that

$$
-(\tfrac{i}{2}+j)g_j^- = (\tfrac{i}{2}-j)g_{i+j}^-. \tag{3.32}
$$

Set $j=0$, we get that $-\frac{i}{2}g_0^-=\frac{i}{2}g_i^-$, i.e., $g_i^-=-g_0^-$ for $i\in\mathbb{Z}\setminus\{0\}$. Set $j=-i$ in (3.32), we get that $-(\frac{i}{2}-i)g_{-i}^-=(\frac{i}{2}+i)g_0^-$, i.e., $g_{-i}^-=\frac{3}{2}g_0^-$ for $i\in\mathbb{Z}\setminus\{0\}$. Then we must have that

$$g_i^-=0 \ \text{ for all } \ i\in\mathbb{Z}. \tag{3.33}$$

Applying $[L_i,G_j^+]=(\frac{i}{2}-j)G_{i+j}^+$ to x_0, we obtain that

$$l_i=-(\tfrac{i}{2}+j)g_j^+-(\tfrac{i}{2}-j)g_{i+j}^+. \tag{3.34}$$

Following $[H_i,G_j^+]\cdot x_0=G_{i+j}^+\cdot x_0$, we get that

$$h_i=g_j^+-g_{i+j}^+ \ \text{ for all } \ i,j\in\mathbb{Z}. \tag{3.35}$$

Set $j=0$, and by (3.31), one can get that

$$g_i^+=g_0^+-ih_1. \tag{3.36}$$

It follows (3.34) that

$$l_i=\tfrac{i^2}{2}h_1-ig_0^+. \tag{3.37}$$

If $h_1=0$, then

$$l_i=-ig_0^+,\quad h_i=0,\quad g_i^-=0,\quad g_i^+=g_0^+,$$

and it satisfies (3.30) (rescaling x_0 by $(g_0^+)^{-1}x_0$). Then it is not a deformation of V. Hence we suppose that $h_1\neq 0$. Rescaling x_0 by $h_1^{-1}x_0$, and follows (3.34), (3.36), we can obtain that

$$l_i=-i\tfrac{g_0^+}{h_1}+\tfrac{i^2}{2},\quad h_i=i,\quad g_i^+=\tfrac{g_0^+}{h_1}-i,\quad g_i^-=0. \tag{3.38}$$

Then we get a deformation of $RA_{a,b}$, denoted by RA_α, which is an indecomposable module with the following relations:

$$RA_\alpha\colon\ \begin{array}{lll} L_ix_k=-(i+k)x_{i+k}, & L_ix_0=(i\alpha+\frac{i^2}{2})x_i, & L_iy_j=-(\frac{i}{2}+j)y_{i+j},\\ H_iy_j=y_{i+j}, & H_ix_k=0, & H_ix_0=ix_i,\\ G_i^-x_j=G_i^+y_j=0, & G_i^-y_j=-2(i+j)x_{i+j}, & \\ G_i^+x_k=y_{i+k}, & G_i^+x_0=-(\alpha+i)y_i, & \end{array} \tag{3.39}$$

where $\alpha=-\frac{g_0^+}{h_1}\in\mathbb{C}$, $j,k\in\mathbb{Z}$ and $k\neq 0$.

Subcase 1.2. There exists $y_{j_0}\in V'$ for some $j_0\in\mathbb{Z}$. By (3.25), in order to make V' is a proper submodule of V, we must have that for all $i\in\mathbb{Z}$,

$$G_i^-y_{j_0}=2(a+i-j_0+2ib)x_{i+j_0}=0,$$

it follows that $a=j_0, b=-\frac{1}{2}$. Without loss of generality, we can suppose that $j_0=0$, then we have $a=0,b=-\frac{1}{2}$. Therefore, $V'=\mathbb{C}y_0$ is a trivial proper

submodule of V, and set $V'' = \mathrm{span}_{\mathbb{C}}\{x_i, y_j \mid i, j \in \mathbb{Z}, j \neq 0\}$. We have the following relations:

$$\begin{array}{llll} L_i x_j &= -(\frac{i}{2}+j)x_{i+j}, & L_i y_k &= -k y_{i+k}, \\ H_i x_j &= -x_{i+j}, & H_i y_k &= 0, \\ G_i^- x_k &= G_i^+ y_j = 0, & G_i^+ x_k = & y_{i+k}, \quad G_i^- y_j = -2j x_{i+j}, \end{array} \tag{3.40}$$

where $j, k \in \mathbb{Z}$ and $k \neq -i$. Suppose that

$$L_i y_{-i} = l_i y_0, \quad H_i y_{-i} = h_i y_0, \quad G_i^{\pm} x_{-i} = g_i^{\pm} y_0.$$

Similar to the arguments about Case 1, one can get that

$$l_i = i g_0^+ + \tfrac{i^2}{2} h_1, \quad g_i^+ = g_0^+ + i h_1, \quad g_i^- = 0, \quad h_i = i h_1.$$

If $h_1 = 0$, then

$$l_i = i g_0^+, \quad g_i^+ = g_0^+, \quad g_i^- = 0, \quad h_i = 0.$$

It is not a deformation too, so we suppose that $h_1 \neq 0$. Rescaling y_0 by $h_1 y_0$, we can get a new representation of $\mathcal{L}$, we denote it by RA^β.

RA^β: satisfies (3.40) and the following relations (set $\beta = \frac{g_0^+}{h_1}$):

$$L_i y_{-i} = (i\beta + \tfrac{i^2}{2}) y_0, \quad H_i y_{-i} = i y_0, \quad G_i^- x_{-i} = 0, \quad G_i^+ x_{-i} = (\beta + i) y_0. \tag{3.41}$$

Obviously, RA_α has a nontrivial submodule with codimention one, and RA^β has a trivial submodule with dimention one.

Case 2. Now we discuss the deformations of $RB_{a,b}$. Since the discussion is similar to Case 1, we will not give the detail, and only enumerate the results. $RB_{a,b}$ also has two deformations:

$$RB_\alpha: \quad \begin{array}{lll} L_i x_k = -(i+k)x_{i+k}, & L_i x_0 = -(i\alpha + \frac{i^2}{2})x_i, & L_i y_j = -(\frac{i}{2}+j)y_{i+j}, \\ H_i y_j = -y_{i+j}, & H_i x_k = 0, & H_i x_0 = i x_i, \\ G_i^+ x_j = G_i^- y_j = 0, & G_i^+ y_j = -2(i+j)x_{i+j}, & \\ G_i^- x_k = y_{i+k}, & G_i^- x_0 = (\alpha + i) y_i, & \end{array} \tag{3.42}$$

where $\alpha = \frac{g_0^-}{h_1} \in \mathbb{C}$, $h_1 \neq 0$, $j, k \in \mathbb{Z}$, and $k \neq 0$.

$$RB^\beta: \quad \begin{array}{lll} L_i x_j = -(\frac{i}{2}+j)x_{i+j}, & L_i y_k = -k y_{i+k}, & L_i y_{-i} = -(i\beta + \frac{i^2}{2}) y_0, \\ H_i x_j = x_{i+j}, & H_i y_k = 0, & H_i y_{-i} = i y_0, \\ G_i^+ x_j = G_i^- y_j = 0, & G_i^- x_k = y_{i+k}, & \\ G_i^- x_{-i} = -(\beta + i) y_0, & G_i^+ y_j = -2j x_{i+j}, & \end{array} \tag{3.43}$$

where $\beta = -\frac{g_0^-}{h_1} \in \mathbb{C}$, $h_1 \neq 0$, $j, k \in \mathbb{Z}$ and $k \neq -i$.

This completes the proof of Theorem 1.1.

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